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Symmetry group of the non-linear Klein–Gordon equation

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Abstract. Maximal symmetry groups for non-linear Klein-Gordon equations of soliton physics have been obtained using Lie's method of extended groups. For general non-linearity, including the sine-Gordon and the double sine-Gordon equation, the Poincaré group is the maximal symmetry group. It has been shown that the symmetry group is larger than the Poincaré group only for power law type non-linearity $V_0\Psi^n$ and exponential type non-linearity $V_0 \exp(-n\Psi)$. For all exponential and power law types with $n \neq 0$, 1, 3 the symmetry group is the Weyl group containing the extra scaling transformation over and above the Poincaré transformations. When $n = 0$ the symmetry group is the semi-direct product of the Abelian group of Ψ translation and the Weyl group. When $n = 1$ the symmetry group is the direct product of the Poincaré group and the Abelian group of Ψ scaling. When $n = 3$ the conformal group is the maximal symmetry group.

1. Introduction

In theoretical analysis of soliton-bearing systems non-linear Klein-Gordon equations occur in diverse circumstances. In the A phase of superfluid liquid ^{3}He the spin waves are governed (after a rescaling) by the sine-Gordon equation (Leggett 1975). Similarly in the B phase the double sine-Gordon is the governing equation (Leggett 1975). Another non-linear case is the Ψ^4 equation which occurs in self-dual Yang-Mills field theory of particle physics (Jackiw 1977) and lattice dynamical studies (Krumhansl and Schrieffer 1975). The fluxon motion in Josephson transmission lines with a distributed bias current is described (Scott 1981) by a two-dimensional (one space and one time) dissipative sine-Gordon equation. The one-dimensional planar ferromagnet like $CsNiF_3$ in an external magnetic field is again described (Mikeska 1978) by the sine-Gordon equation.

In the above-mentioned analyses interest is centred around the soliton-like solutions of the equations. Here we analyse the group structure of the non-linear Klein-Gordon equation. The largest symmetry group that keeps the equation invariant gives a deeper insight into the totality of the solutions. To these ends we have used Lie's extended group method (Dickson 1924, Hamermesh 1984, Hill 1982, Sattinger 1977, Rudra 1984) to obtain the generators of the maximal symmetry group of the non-linear Klein-Gordon equation. It is found that for general non-linearity including both the sine-Gordon and the double sine-Gordon equations, the maximal symmetry group is the Poincaré group. Only if the non-linearity is either of the exponential form $V_0 \exp(-n\Psi)$ or the power law form $V_0\Psi^n$, is the symmetry group a larger group containing the Poincaré group as a proper subgroup. For the exponential type and the power law type with the index $n \neq 0$, 1, 3 non-linearity the Weyl group is the symmetry group. For $n = 0$ the maximal symmetry group is the semi-direct product of the Abelian group of translation in the Ψ space and the Weyl group. For $n=1$ the maximal symmetry group is the direct product of the Poincaré group and the one-parameter Abelian group of Ψ scaling. The case with $n=3$ has the conformal group (Fulton *et al* 1962, Wess 1960) containing the Weyl group and the four-parameter Abelian group of acceleration transformations as the maximal symmetry group.

2. Lie's extended group method

We first give a short description (Hamermesh 1984, Hill 1982, Sattinger 1977, Rudra 1984) of Lie's extended group method and the procedure for obtaining the maximal symmetry group of a set $\Delta^\alpha(q, \Psi; k)=0$, $\alpha=1, \ldots, p$ of partial differential equations. Here k is the highest order of partial derivatives of Ψ appearing in the set, q_i with $i=1, \ldots, n$ are the independent variables and Ψ^m with $m=1, \ldots, r$ are the dependent variables. We first construct a space encompassing all the variables and the derivatives q_i, Ψ^m, Ψ_J^m where

$$\Psi_J^m=\partial^{|J|}\Psi^m\left(\prod_{i=1}^n (\partial q_i)^{j_i}\right) \qquad \text{with } J\equiv(j_1, j_2, \ldots, j_n),\ |J|=\sum_i j_i. \tag{1}$$

Here j_i are non-negative integers. If

$$X=\sum_i \xi^i(q, \Psi)\,\partial/\partial q_i+\sum_m \varphi_m(q, \Psi)\,\partial/\partial\Psi^m \tag{2}$$

is the generator of the product space (q, Ψ) then the kth extension $X^{(k)}$ of X is given by

$$X^{(k)}=X+\sum_m \sum_{1\leqslant|J|\leqslant k} \varphi_m^J\,\partial/\partial\Psi_J^m. \tag{3}$$

Here

$$\varphi_m^J=D^J\left(\varphi_m-\sum_i \Psi_i^m\,\xi^i\right)+\sum_i \Psi_{J,i}^m\xi^i \tag{4}$$

where

$$\Psi_i^m=\partial\Psi^m/\partial q_i \qquad (J, i)\equiv(j_1, \ldots, j_{i-1}, j_i+1, j_{i+1}, \ldots, j_n) \tag{5}$$

and

$$D^J=D_1^{j_1}\ldots D_n^{j_n} \qquad D_i=\partial/\partial q_i+\sum_m \sum_{0\leqslant|J|\leqslant k} \Psi_{J,i}^m\,\partial/\partial\Psi_J^m. \tag{6}$$

The system of partial differential equations

$$\Delta^\alpha(q, \Psi; k)=0 \qquad \alpha=1, \ldots, p \tag{7}$$

has the maximal symmetry group G if for every generator $X\in$ G

$$X^{(k)}\Delta^\alpha(q, \Psi; k)=0 \qquad \alpha=1, \ldots, p. \tag{8}$$

On the left-hand side of (8) we use (7) and equate to zero the coefficients of the different-order partial derivatives of Ψ^m and their powers and products. This process gives us a set of partial differential equations for ξ and Ψ. Their solutions give us the most general form of X and hence the maximal symmetry group G. It should be noted that in (2)-(6) and (8) q_i, Ψ_m and Ψ_J^m are to be treated as independent variables.

3. Non-linear Klein–Gordon equation

We now treat the general non-linear Klein–Gordon equation

$$\sum_{\alpha} \Psi_{\alpha\alpha} - \Psi_{\tau\tau} + V(\Psi) = 0. \tag{9}$$

Here $\tau = vt$ is the reduced time variable and q_α ($\alpha = 1, 2, 3$) are the cartesian components of the space variable. A Greek variable as a subscript, except with q, denotes the corresponding partial derivative. We write

$$X = \xi^\tau \, \partial/\partial\tau + \sum_\alpha \xi^\alpha \, \partial/\partial q_\alpha + \varphi \, \partial/\partial\Psi$$

and obtain from (1)–(8) the following partial differential equations for ξ and φ:

$$\begin{aligned} &\xi^\tau_\Psi = \xi^\alpha_\Psi = 0 \qquad \varphi_{\Psi\Psi} = 0 \qquad \xi^\alpha_\beta + \xi^\beta_\alpha = 0 \quad (\alpha \neq \beta) \qquad \xi^\alpha_\tau - \xi^\tau_\alpha = 0 \\ &\xi^\alpha_\alpha = \xi^\tau_\tau \qquad \sum_\alpha \xi^\alpha_{\alpha\alpha} - \xi^\alpha_{\tau\tau} = 2\varphi_{\Psi\alpha} \qquad \sum_\alpha \xi^\tau_{\alpha\alpha} - \xi^\tau_{\tau\tau} = 2\varphi_{\Psi\tau} \\ &\sum_\alpha \varphi_{\alpha\alpha} - \varphi_{\tau\tau} + (2\xi^\tau_\tau - \varphi_\Psi) V(\Psi) + \varphi V'(\Psi) = 0. \end{aligned} \tag{10}$$

Here the prime on V means derivative with respect to Ψ.

The first six equations of (10) have the general solution

$$\begin{aligned} \xi^\alpha &= a_\alpha + bq_\alpha + \sum_{\beta\gamma} e_{\alpha\beta\gamma} b^\gamma q_\beta + b_{3+\alpha}\tau - C_\alpha q^2 + 2q_\alpha\left(\sum_\beta C_\beta q_\beta + C_4\tau\right) \\ \xi^\tau &= a_4 + b\tau + \sum_\beta b_{3+\beta} q_\beta + C_4 q^2 + 2\tau\left(\sum_\beta C_\beta q_\beta + C_4\tau\right) \\ \varphi &= A(q, \tau) + 2\left(C_0 - \sum_\beta C_\beta q_\beta - C_4\tau\right)\Psi \end{aligned} \tag{11}$$

with

$$q^2 = \sum_\beta q_\beta^2 - \tau^2.$$

Here a, b and C are constants. The last equation of (10) gives

$$\begin{aligned} &\sum_\alpha A_{\alpha\alpha} - A_{\tau\tau} + 2\left(b - C_0 + 3\sum_\alpha C_\alpha q_\alpha + 3C_4\tau\right) V(\Psi) \\ &\qquad + \left[A + 2\Psi\left(C_0 - \sum_\alpha C_\alpha q_\alpha - C_4\tau\right)\right] V'(\Psi) = 0. \end{aligned} \tag{12}$$

We note that the set of generators originating from a_α, a_4, b^γ and $b_{3+\alpha}$ are independent of the set originating from b, C_α, C_4 and $A(q, \tau)$. The former set, being independent of the form of the non-linearity $V(\Psi)$, must be present in the maximal symmetry group of the non-linear Klein–Gordon equation for all types of non-linearity. These ten generators

$$\begin{aligned} &X^\alpha = -\mathrm{i}\,\partial/\partial q_\alpha \qquad X^\tau = -\mathrm{i}\,\partial/\partial\tau \qquad X^\alpha_{\mathrm{R}} = -\mathrm{i}\sum_{\beta\gamma} e_{\alpha\beta\gamma} q_\beta \, \partial/\partial q_\gamma \\ &X^\alpha_{\mathrm{L}} = \tau\,\partial/\partial q_\alpha + q_\alpha \, \partial/\partial\tau \end{aligned} \tag{13}$$

with the non-vanishing commutation relations

$$\begin{array}{lll} [X^\alpha, X_R^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X^\gamma & [X^\alpha, X_L^\beta] = \delta_{\alpha\beta} X^\tau & [X^\tau, X_L^\alpha] = X^\alpha \\ [X_R^\alpha, X_R^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_R^\gamma & [X_R^\alpha, X_L^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_L^\gamma & [X_L^\alpha, X_L^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_R^\gamma \end{array} \tag{14}$$

where $e_{\alpha\beta\gamma}$ is the permutation symbol, form the Poincaré group. The first four generators are those for translations along the q_α and τ axes, X_R^α are those for rotations in the q_α space and X_L^α are those for the Lorentz boosts.

If this group is the full symmetry group and there are no other generators then we must have $A = b = C_0 = C_\alpha = C_4 = 0$. We now investigate if the symmetry group can be larger than the Poincaré group and find all the potentials $V(\Psi)$ for which this is the case. This will happen if at least one of the quantities $A(q, \tau)$, b, C_0, C_α and C_4 is non-zero. Differentiating (12) with respect to Ψ we get

$$2\left(b + 2\sum_\alpha C_\alpha q_\alpha + 2C_4\tau\right) V'(\Psi) + \left[A(q,\tau) + 2\Psi\left(C_0 - \sum_\alpha C_\alpha q_\alpha - C_4\tau\right)\right] V''(\Psi) = 0. \tag{15}$$

$$V''(\Psi)/V'(\Psi) = -2\left(b + 2\sum_\alpha C_\alpha q_\alpha + 2C_4\tau\right)\left[A(q,\tau) + 2\Psi\left(C_0 - \sum_\alpha C_\alpha q_\alpha - C_4\tau\right)\right]^{-1} \tag{16}$$

must be a function of Ψ only and thus independent of q_α and τ. This can happen if $A(q, \tau)$ is a constant A, zero or non-zero.

3.1. $A = 0$

There are two cases here; either the coefficient of Ψ in the denominator of the right-hand side of (16) is a constant multiple of the numerator or $C_\alpha = C_4 = 0$.

(i) $$\left(b + 2\sum_\alpha C_\alpha q_\alpha + 2C_4\tau\right) = k\left(C_0 - \sum_\alpha C_\alpha q_\alpha - C_4\tau\right) \tag{17}$$

where k is a constant. Since (17) must hold for all values of q_α and τ, we have $k = -2$ and $C_0 = -b/2$. Equation (12) now becomes $3V(\Psi) - \Psi V'(\Psi) = 0$ yielding $V(\Psi) = V_0\Psi^3$. In this case the maximal symmetry group has five more generators over and above those of the Poincaré group:

$$\begin{array}{ll} X_0 = \sum_\alpha q_\alpha\, \partial/\partial q_\alpha + \tau\, \partial/\partial\tau - \Psi\, \partial/\partial\Psi & \\ X_A^\alpha = -iq^2 X^\alpha/2 + q_\alpha X_0 & X_A^\tau = iq^2 X^\tau/2 + \tau X_0. \end{array} \tag{18}$$

The extra non-vanishing commutation relations are

$$\begin{array}{lll} [X^\alpha, X_0] = X^\alpha & [X^\alpha, X_A^\beta] = -i\delta_{\alpha\beta}X_0 - \sum_\gamma e_{\alpha\beta\gamma} X_R^\gamma & [X^\alpha, X_A^\tau] = -iX_L^\alpha \\ [X^\tau, X_0] = X^\tau & [X^\tau, X_A^\alpha] = -iX_L^\alpha & [X^\tau, X_A^\tau] = -iX_0 \\ [X_R^\alpha, X_A^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_A^\gamma & [X_L^\alpha, X_A^\beta] = \delta_{\alpha\beta} X_A^\tau & [X_L^\alpha, X_A^\tau] = X_A^\alpha \\ [X_0, X_A^\alpha] = X_A^\alpha & [X_0, X_A^\tau] = X_A^\tau. & \end{array} \tag{19}$$

The generator X_0 corresponds to the scaling transformation $q'_\alpha = sq_\alpha$, $\tau' = s\tau$, $\Psi' = \Psi/s$. The last four generators form the four-parameter Abelian group of acceleration transformations (Fulton *et al* 1962, Wess 1960) corresponding to the point transformations

$$q'_\alpha = (q_\alpha + a_\alpha q^2)/(1+2aq+a^2q^2)$$
$$\tau' = (\tau + a_4 q^2)/(1+2aq+a^2q^2) \qquad \Psi' = \Psi(1+2aq+a^2q^2)$$

with

$$aq = \sum_\alpha a_\alpha q_\alpha - a_4\tau \qquad a^2 = \sum_\alpha a_\alpha^2 - a_4^2. \tag{20}$$

This 15-parameter group is the conformal group (Fulton *et al* 1962, Wess 1960) which was shown by Bateman (1910) to be the group of Maxwell's electromagnetic equations.

(ii) Referring to (12) we find

$$\Psi V'(\Psi)/V(\Psi) = \left(b - C_0 + 3\sum_\alpha C_\alpha q_\alpha + 3C_4\tau\right)\left(C_0 - \sum_\alpha C_\alpha q_\alpha - C_4\tau\right)^{-1}.$$

In order that the right-hand side is independent of q_α we have the relation $C_4 = C_\alpha = 0$ with $C_0 = -b/(n-1)$ thus giving $V(\Psi) = V_0\Psi^n (n \neq 1, 3)$. This gives the extra generator

$$X_0 = \sum_\alpha q_\alpha\, \partial/\partial q_\alpha + \tau\, \partial/\partial\tau - [2\Psi/(n-1)]\, \partial/\partial\Psi \tag{21}$$

with the extra non-vanishing commutator

$$[X^\alpha, X_0] = X^\alpha \qquad [X^\tau, X_0] = X^\tau \tag{22}$$

over and above those given in (13) and (14). This generator corresponds to the scaling transformation $q'_\alpha = sq_\alpha$, $\tau' = s\tau$, $\Psi' = \Psi/s^{2/(n-1)}$. The maximal symmetry group is thus isomorphic to the Weyl group.

3.2. $A(q, \tau) = A = constant \neq 0$

Referring to (16) we see that in this case $C_4 = C_\alpha = 0$. Thus (12) now becomes

$$2(b - C_0)V(\Psi) + (A + 2C_0\Psi)V'(\Psi) = 0. \tag{23}$$

(i) If both b, $C_0 = 0$, then $V(\Psi) = V_0 =$ constant corresponding to the inhomogeneous Klein–Gordon equation. We now have the extra generator $X^\Psi = -\mathrm{i}\, \partial/\partial\Psi$ commuting with all the ten generators of (13). This generator, of course, corresponds to the translation in the Ψ space $q'_\alpha = q_\alpha$, $\tau' = \tau$, $\Psi' = \Psi + s$.

(ii) If $C_0 = 0$ and $b \neq 0$ then $V(\Psi) = V_0 \exp(-n\Psi)$, where $A = 2b/n$ and $n \neq 0$. The extra generator that arises is

$$X_0 = \sum_\alpha q_\alpha\, \partial/\partial q_\alpha + \tau\, \partial/\partial\tau + (2/n)\, \partial/\partial\Psi \tag{24}$$

with the non-vanishing commutators of (22). This generator corresponds to the transformation $q'_\alpha = sq_\alpha$, $\tau' = s\tau$, $\Psi' = \Psi + (2/n) \ln s$. The maximal symmetry group is again isomorphic to the Weyl group.

(iii) If $b = 0$ and $C_0 \neq 0$, then $V(\Psi) = V_0(m + \Psi)$, where $A = 2mC_0$. By a suitable transformation $\Psi \to m + \Psi$ we can make this case equivalent to the linear equation with $V(\Psi) = V_0\Psi$. The extra generator now is $X_0 = \Psi\, \partial/\partial\Psi$ commuting with all the ten generators of the Poincaré group. This generator corresponds to the transformation $q'_\alpha = q_\alpha$, $\tau' = \tau$, $\Psi' = s\Psi$. The maximal symmetry group here is the direct product of the Poincaré group and the Abelian group of Ψ scaling.

(iv) If b, $C_0 \neq 0$, then $V(\Psi) = V_0(m+\Psi)^n$, where $C_0 = -b/(n-1)$ and $A = -2bm/(n-1)$ with $n \neq 1$. By the transformation $\Psi \to m+\Psi$ this case reduces to case (ii) in § 3.1 with $V(\Psi) = V_0\Psi^n$. We note that for $n=0$ this case becomes the inhomogeneous Klein-Gordon equation with $V(\Psi) = V_0$, and thus has both the extra generators X^Ψ of case (i) in this subsection and X_0 of (21) with $n=0$. The commutation relation between X^Ψ and X_0 is $[X^\Psi, X_0] = 2X^\Psi$. The maximal symmetry group is thus the semi-direct product of the Abelian group of Ψ translation and the Weyl group.

The only other possibility that $A(q, \tau)$ and $b + 2\sum_\alpha C_\alpha q_\alpha + 2C_4\tau$ are constant multiples of $C_0 - \sum_\alpha C_\alpha q_\alpha - C_4\tau$ is valid for the potential $V(\Psi) = V_0(m+\Psi)^3$ which reduces to the case $V(\Psi) = V_0\Psi^3$ when we make the transformation $\Psi \to m+\Psi$. In all other cases of non-linearity the Poincaré group is the maximal symmetry group.

It should be mentioned that the groups obtained here are the connected components of the identity element. Except for the odd-indexed power law type non-linearity there are four pieces corresponding to the discrete transformations $q'_\alpha = -q_\alpha$, $\tau' = -\tau$. For the odd-indexed power law there is another piece corresponding to $\Psi' = -\Psi$.

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